

## Section 6.8 Indeterminate Forms and L'Hospital's Rule.

If we were to evaluate the function  $f(x) = \frac{\ln x}{x-1}$  at  $x=1$ , we'd be in trouble, since  $f(1) = \frac{\ln 1}{1-1} = \frac{0}{0}$ ! Alternatively, we may ask: what is  $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$ ?

In general, if  $f(x) \rightarrow 0$  and  $g(x) \rightarrow 0$  as  $x \rightarrow a$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  may or may not exist, and is called an indeterminate form of type  $\frac{0}{0}$ .

Note that we have encountered this type before:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{x}{x+1} = \frac{1}{2}, \text{ etc.}$$

For these examples, we had tricks to find the limits. Now we seek a method that works every time.

Another problem with  $f(x) = \frac{\ln x}{x-1}$  is in Evaluating its limit at  $+\infty$ .

In general, if  $f(x) \rightarrow \pm\infty$  and  $g(x) \rightarrow \pm\infty$  as  $x \rightarrow a$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  may or may not exist, and is called an indeterminate form of type  $\frac{\infty}{\infty}$ .

We have also encountered this type before:

$$\lim_{x \rightarrow \infty} \frac{3x^3 - x}{2x^3 + 1} = \lim_{x \rightarrow \infty} \frac{3 - \frac{1}{x^2}}{2 + \frac{1}{x^3}} = \frac{3}{2}. \quad \text{Here we used a trick. Again, we want a method that works every time.}$$

L'Hospital's Rule: suppose  $f$  and  $g$  are differentiable functions, and  $g'(x) \neq 0$  on an open interval  $I$  that contains  $a$  (except possibly at  $a$ ); suppose that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0, \text{ or } \lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty. \text{ Then}$$

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  if the right hand side limit exists, or if it is equal to  $+\infty$  or  $-\infty$ .

Remarks: - you must always check the conditions of L'Hospital's Rule before using it.

- L'Hospital's Rule works for one-sided limits and for limits at  $+\infty$  and  $-\infty$  as well.
- For a proof of special cases of L'Hospital's Rule, see your book or come see me.

Examples ①  $\lim_{x \rightarrow 1} \frac{\ln x}{x-1} \left( \frac{0}{0} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow 1} \frac{\frac{1}{x} \ln x}{\frac{1}{x}(x-1)} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} = 1.$

②  $\lim_{x \rightarrow \infty} \frac{e^{3x} - 1}{x^2 + x - 1} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow \infty} \frac{3e^{3x}}{2x+1} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow \infty} \frac{9e^{3x}}{2} = \infty.$

③  $\lim_{x \rightarrow \infty} \frac{\ln(2x)}{\sqrt[4]{x}} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{4}x^{-\frac{3}{4}}} = \lim_{x \rightarrow \infty} \frac{4x^{\frac{3}{4}}}{x} = \lim_{x \rightarrow \infty} \frac{4}{x^{\frac{1}{4}}} = 0.$

④  $\lim_{x \rightarrow 1} \frac{\sin(\frac{\pi}{2}x) - 1}{x-1} \left( \frac{0}{0} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow 1} \frac{\frac{\pi}{2}\cos(\frac{\pi}{2}x)}{1} = \frac{\pi}{2} \cdot 0 = 0.$

⑤  $\lim_{x \rightarrow \infty} \frac{\ln x}{x-1} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$

• Indeterminate products occur when  $\lim_{x \rightarrow a} f(x) \cdot g(x)$  has the form  $0 \cdot \infty$

Examples ①  $\lim_{x \rightarrow 0^+} x \cdot \ln x \quad (0 \cdot (-\infty)) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0.$

②  $\lim_{x \rightarrow -\infty} x e^x \quad (-\infty \cdot 0) = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = \frac{1}{\infty} = 0.$

• Indeterminate differences occur when  $\lim_{x \rightarrow a} (f(x) - g(x))$  has the form  $\infty - \infty$ .

In this case, we should try to rewrite  $f(x) - g(x)$  as a quotient.

Example  $\lim_{x \rightarrow 0} \csc(x) - \cot(x) \quad (\infty - \infty) = \lim_{x \rightarrow 0} \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} \left( \frac{0}{0} \right)$   
 $\xrightarrow{\text{L.H.}} \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0.$

. Indeterminate Powers occur when  $\lim_{x \rightarrow a} (f(x))^{g(x)}$  has the form  $0^0$ ,  $\infty^\infty$ , or  $1^\infty$ .

We deal with these cases by using the natural log.

Example ①  $\lim_{x \rightarrow 0^+} (x^x)$  (form  $0^0$ ). Let  $y = x^x$ , then  $\ln y = x \cdot \ln x$ .

$$\lim_{x \rightarrow 0^+} \ln(y) = \lim_{x \rightarrow 0^+} x \cdot \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \left( \frac{-\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

Since  $\lim_{x \rightarrow 0^+} \ln(y) = 0$ , then  $\lim_{x \rightarrow 0^+} e^{\ln(y)} = e^0 \Rightarrow \lim_{x \rightarrow 0^+} y = 1$ .

②  $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x}$  (form  $1^\infty$ ). Let  $y = (1 + \sin 4x)^{\cot x}$ ; then we have

$$\ln(y) = \cot x \cdot \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}. \text{ We calculate}$$

$$\lim_{x \rightarrow 0^+} \ln(y) = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} \left( \frac{0}{0} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow 0^+} \frac{\frac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = \frac{4}{1} = 4. \text{ Therefore } \lim_{x \rightarrow 0^+} y = e^4.$$

③  $\lim_{x \rightarrow \infty} (e^x)^{\sin(\frac{1}{x})}$  (form  $\infty^0$ ). Let  $y = (e^x)^{\sin(\frac{1}{x})}$ ; then we have

$$\ln(y) = \sin(\frac{1}{x}) \ln(e^x) = x \cdot \sin(\frac{1}{x}) = \frac{\sin(\frac{1}{x})}{\frac{1}{x}}. \text{ We calculate}$$

$$\lim_{x \rightarrow \infty} \ln(y) = \lim_{x \rightarrow \infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} \left( \frac{0}{0} \right) \xrightarrow{\text{L.H.}} \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} \cos(\frac{1}{x})}{(-\frac{1}{x^2})} = \cos(0) = 1$$

Therefore  $\lim_{x \rightarrow \infty} y = e^1 = e$ .